9/19/25 Lecture 12

* Sometimes multiplying matrices as opposed to doing many matrix times vector operations can reduce the number of total operations.

Definition : A matrix A is invertible if $\exists$ matrix B such that $AB = BA = I$
We call B the inverse of A and write $B = A^{-1}$

Example : $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

verify: $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

How to take advantage of A^{-1} ?

Consider $\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 5 & 3 & 6 \end{array}\right] = \left[\begin{array}{c|c} A & \begin{matrix} 1 \\ 6 \end{matrix} \end{array}\right]$

ie : $A\vec{x} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

multiply both sides by A^{-1}

$$A^{-1} A(\vec{x}) = A^{-1} \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

$$\underbrace{(A^{-1} A)}(\vec{x}) = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

$$I\, (\vec{x}) = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$$

Theorem: Suppose A is an invertible $m \times n$ matrix. Then $\forall\ \vec{b} \in \mathbb{R}^m$ the linear system $A\vec{x} = \vec{b}$ has exactly one solution $\vec{x} = A\vec{b}$.

Proof: If $\vec{x}$ solves $A\vec{x} = \vec{b}$ then $A^{-1}(A\vec{x}) = A^{-1}\vec{b}$

$$\hookrightarrow (A^{-1}A)\vec{x} = A^{-1}\vec{b}$$

$$\hookrightarrow I\vec{x} = A^{-1}\vec{b}$$

$$\hookrightarrow \vec{x} = A^{-1}\vec{b}$$

OTOH: If $\vec{x} = A^{-1}\vec{b}$, then

$$\begin{aligned} A\vec{x} &= A(A^{-1}\vec{b}) \\ &= (AA^{-1})\vec{b} \\ &= I\vec{b} \\ &= \vec{b} \end{aligned}$$

So $\vec{x} = A^{-1}\vec{b}$ solves the system.

$\therefore$ $A\vec{x} = \vec{b}$ has exactly one solution.

$$\vec{x} = A^{-1}\vec{b}$$

$\blacksquare$

How to find A^{-1}?

Suppose $A = n \times n$ matrix. Then A^{-1} (if it exists) is also $n \times n$.

$A^{-1} = [\vec{x}_1 \ \dots \ \vec{x}_n]$ column-wise

We want $AA^{-1} = I$

i.e. $[A\vec{x}_1 \ \dots \ A\vec{x}_n] = [\vec{e}_1 \ \dots \ \vec{e}_n]$

So to find the columns $\vec{x}_j$ of A^{-1}, I need to solve:

$$A\vec{x}_j = \vec{e}_j \quad \text{or} \quad [A \mid \vec{e}_j]$$

So if $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ then $A^{-1} = [\vec{x}_1 \ \vec{x}_2]$

and $A\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $A\vec{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\tfrac{1}{2}\cdot \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right]$$

$$\to \begin{array}{c} \\ -5R_1 \end{array} \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right]$$

$$\to \begin{array}{c} -R_2 \\ \cdot 2 \end{array} \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & -5/2 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \vec{x}_1 & \vec{x}_2 \end{array}$$

To find A^{-1}, I begin with the augmented matrix

$$[A \mid I]$$

and then do Row Reduction to get A into RREF (ideally I)

$$[A \mid I] \to [I \mid B]$$

eg. $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 1 & 4 \end{bmatrix}$ Find A^{-1}

Solve

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ -1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} -1 & 1 & 4 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} \cdot(-1) \\ \\ \cdot\left(\frac{1}{2}\right) \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -1 & -4 & 0 & 0 & -1 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right] \begin{array}{l} +4(R_3) \\ -3(R_3) \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right] + R_2$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 1 & -1 \\ 0 & 1 & 0 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 1 & -1 \\ -\frac{3}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$